Journal of Pure and Applied Algebra 38 (1985) 213-216
North-Holland

THE ENVELOPING ALGEBRA OF A GRADED LIE ALGEBRA OF GLOBAL DIMENSION TWO CONTAINS A FREE SUBALGEBRA ON TWO GENERATORS

Rikard BØGVAD
Department of Mathematics, MIT, Cambridge, MA 02139, USA

Communicated by C. Löfwall
Received 1 February 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

Introduction

Let $\mathfrak{g} = \bigoplus_{j>0} \mathfrak{g}_j$ be a graded Lie algebra over a field **k** [1, p. 18]; e.g. let $\mathfrak{g}$ be the rational homotopy Lie algebra $\pi^*(\Omega S) \otimes \mathbb{Q}$ of a simply connected topological space S.

The Avramov–Felix conjecture [1, p. 111], in its topological version, says that the rational homotopy Lie algebra of a space of finite rational Lusternik–Schnirelmann category is either finite dimensional as a vector space, or contains a free Lie subalgebra on two generators.

When the L.S. category is one, the conjecture is trivially true, since the entire Lie algebra is then free. In category two there are two cases; when the space is coformal and when it is not. The case where the space is not coformal is treated in [5] and [6], there is a canonical way to find a free Lie subalgebra, and the conjecture is true in this case.

A coformal space of L.S. category two is a space such that the enveloping algebra of the rational homotopy Lie algebra has global dimension two. For such a space, we prove a weaker form of the Avrmaov–Felix conjecture.

Theorem 1. *Let* $\mathfrak{g} = \bigoplus_{j>0} \mathfrak{g}_j$ *be a graded Lie algebra over a field* **k**. *If its enveloping algebra* $U(\mathfrak{g})$ *has global dimension two, then either* $\mathfrak{g}$ *is finite dimensional as a vector space, or* $U(\mathfrak{g})$ *contains a free subalgebra on two homogeneous generators.*

The idea behind the proof is that the condition of having no free subalgebras translates into the nice ring-theoretical property of having a classical quotient field. We hope that this condition is also easier to work with for coformal spaces of higher category. Hence, the weaker form of the Avramov–Felix conjecture corresponding to Theorem 1 might be more accessible.

1. Reduction to the case of one relation

We will consider $\mathfrak{g}_{\text{even}} = \bigoplus_{j>0} \mathfrak{g}_{2j}$, and for this we need two lemmas.

Lemma 1. *Let* $\mathfrak{g} = \bigoplus_{j>0} \mathfrak{g}_{2j}$ *be a graded Lie algebra. if* $U(\mathfrak{g})$ *does not contain a free subalgebra on two homogenous generators, then it has a classical quotient field.*

Proof. Since $\mathfrak{g}$ is concentrated in even degrees, it is a classical (ungraded) Lie algebra, and so $U(\mathfrak{g})$ has no zero-divisiors. But since $U(\mathfrak{g})$ has no free subalgebra on two homogenous generators, it satisfies the Ore conditions [7] for homogenous elements. If a and b are two homogenous elements, there is then a relation between them which can be written $ac = bd$ for some non-zero homogenous elements c and d. Thus, it is possible to invert every non-zero homogenous element. The resulting graded field we denote by K.

Lemma 2. *Let* $\mathfrak{g} = \bigoplus_{j>0} \mathfrak{g}_{2j}$ *be a finitely generated graded Lie algebra such that* $U(\mathfrak{g})$ *has global dimension two. If* $U(\mathfrak{g})$ *does not contain a free subalgebra on two homogenous generators, then* $\mathfrak{g}$ *is finitely presented and, in a minimal presentation, the number of relations is one less than the number of generators.*

Proof. Let $0 \to F_2 \to F_1 \to U(\mathfrak{g}) \to \mathbf{k} \to 0$ be a minimal free resolution of $\mathbf{k}$ over $U(\mathfrak{g})$. The ranks of F_2 and F_1 are the number of relations and generators, respectively. Tensor the exact sequence with K. It is still exact, and we have $\mathbf{k} \otimes_{U(\mathfrak{g})} K = 0$ since $\mathbf{k}$ is annihilated by every homogenous element of positive degree. The rank of free K-modules is well defined since K is a graded field. Thus, it only remains to count the ranks.

Proof of Theorem 1. Assume $U(\mathfrak{g})$ has no free subalgebra on two homogenous generators. Consider a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\text{even}}$ generated by two linearly independent homogenous elements. Clearly, $U(\mathfrak{h})$ has global dimension two and has no free sublagebra. Thus, by Lemma 2, $U(\mathfrak{h})$ has one relation only. In the next section we prove that $U(\mathfrak{h})$ is then commutative. Hence, $\mathfrak{g}_{\text{even}}$ is abelian. Since the global dimension is at most two, $\mathfrak{g}_{\text{even}}$ has dimension at most two as a vector space. In [4] it is proved that the dimension of $\mathfrak{g}_{\text{odd}}$ is then also not greater than two, and thus $\mathfrak{g}$ is finite dimensional as a vector space. In fact, if $\mathfrak{g}$ is generated by elements of degree one, and $\mathbf{k}$ is algebraically closed, then $\mathfrak{g}$ is the product of two free Lie algebras on one generator each.

2. The case of one relation

To prove Theorem 1, it remains to consider $U(\mathfrak{h})$, where $\mathfrak{h}$ is generated by two homogenous elements of even degree, and has one relation only. We prove $U(\mathfrak{h})$ is then commutative; i.e., $\mathfrak{h}$ is abelian.

Proposition 1. *Let $U(\mathfrak{h})$ be given by $U(\mathfrak{h}) = \mathbf{k}\langle x, y\rangle/(r)$, where x and y are homogenous elements of even degree, and where r is a homogenous Lie element. If $U(\mathfrak{h})$ does not contain a free subalgebra on two homogenous generators, then $U(\mathfrak{h})$ is commutative.*

Proof. Let $<$ be the ordering of the monomials of $\mathbf{k}\langle x, y\rangle$ defined by $M<N$ if the length of M is less than the length of N, and by lexicographical ordering of monomials of the same length. Let $h(r)$ be the highest term of r in this ordering.

If $h(r)$ is combinatorially free as a one-element set (cf. [2]), then by [2, Theorem 1.4 and 2.6], two homogenous elements of $\mathbf{k}\langle x, y\rangle$ that generate a free sublagebra of $\mathbf{k}\langle x, y\rangle/(h(r))$ also generate a free subalgebra of $U(\mathfrak{h})$.

We use a property of Lie elements [8, p. 15] to show that $h(r)$ is combinatorially free. When r is a Lie element, $h(r)$ is a monomial with the following property; if $h(r) = uv$ for some non-trivial monomials u and v, then $uv > vu$.

Assume $h(r)$ is not combinatorially free; i.e., there are non-trivial factorizations $h(r) = ab = bc$. It is easy to see that there are then elements m and n, such that $a = mn$, $b = (mn)^i m$ for some integer i, and $c = nm$. Using this,

$$h(r) = ((mn)^i m)(nm) = uv > vu = (nm)((mn)^i m),$$

and so $mn > nm$. But also

$$h(r) = (mn)((mn)^i m) = u'v' > v'u' = ((mn)^i m)(mn),$$

and hence $nm > mn$, which is a contradiction.

It only remains to use [3], which contains complete results on free subalgebras of rings with monomial relations. In our case, by [3], $\mathbf{k}\langle x, y\rangle/(h(r))$, and hence $U(\mathfrak{h})$, contains a free subalgebra if $h(r)$ is not xy or yx. But when $h(r)$ is xy or yx, then $r = [x, y]$, and so $U(\mathfrak{h})$ is commutative, which was to be proved.

Acknowledgements

The author would like to thank D. Anick, C. Löfwall, and C. Jacobsson for helpful discussions.

References

[1] J.-M. Lemaire and J.-C. Thomas, eds., Homotopie Algébrique et Algèbre Locale, Astérisque 113–114 (1984).

[2] D. Anick, Non-commutative graded algebras and their Hilbert series, J. Algebra 78 (1982) 120–140.

[3] D. Anick, Free subalgebras of graded algebras, to appear.

[4] R. Bøgvad, Some elementary results on the cohomology of graded Lie algebras, in: [1], pp. 156–166.

[5] Y. Felix and J.-C. Thomas, Characterization of spaces whose rational L.S. category is two, Illinois J. Math., to appear.

[6] C. Löfwall, On the subalgebra generated by the one-dimensional elements of the Yoneda Ext-algebra, in: J.-E. Roos, ed., Algebra, Algebraic Topology, and their Interactions, Lecture Notes in Math. (Springer, Berlin, to appear).
[7] B. Stenström, Rings of Quotients, Grundlehren 217 (Springer, Berlin, 1975).
[8] G. Viennot, Algèbres de Lie libres, Lecture Notes in Math. 691 (Springer, Berlin, 1978).